# Motivic Aspects of Mixed Hodge structures

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We outline the proof that the Hodge-characteristic (when defined via graded pieces of a mixed Hodge structure) is a ring homomorphism $\mathrm{K}_0(\mathrm{Var}) \to \mathrm{K}_0(\mathfrak{hs})$, as in [Pet10, Lec.6]. Along the way, we outline some problems relating to triangulated structures on the category of complexes with mixed Hodge structures. These notes were prepared for the Gradutate Seminar on Motivic Hodge theory, which took place in Bonn during the summer semester 2023. I would like to thank both Prof. Daniel Huybrechts and Marco Volpe for helping me prepare for this talk. Some claims and questions I could not resolved are marked in pink.

## Contents



## 1 Recap and overview

**1.1.** Recall our setup: Let Var be the category of complex algebraic varieties. We have the Grothendieck group of varities $\mathrm{K}_0(\mathrm{Var}) \coloneqq \mathbb{Z}[\mathrm{Var}]/J$, where $J$ is the ideal generated by the scissor relations

$$[X] - [Y] - [X \setminus Y] = 0 \tag{1}$$

for any inclusion $Y \subseteq X$ of closed subvarieties. We already know that there exists a unique ring homomorphism

$$\chi^{\mathrm{Hdg}} \colon \mathrm{K}_0(\mathrm{Var}) \to \mathrm{K}_0(\mathfrak{hs}) \tag{2}$$

called the *Hodge characteristic*, that agrees with the topological Euler characteristic after applying the natural map $\mathrm{K}_0(\mathfrak{hs}) \to \mathrm{K}_0(\mathrm{Vec})$, and that satisfies

$$\chi^{\mathrm{Hdg}}(X) = \sum (-1)^k \, \mathrm{H}^k(X)$$

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*whenever $X$ is smooth projective.* One goal is now to give an explicit description of $\chi^{\mathrm{Hdg}}(X)$ for arbitrary $X$. This will be done using the language of mixed Hodge structures:

*Definition 1.1a — Let $H$ be a $\mathbb{Q}$-module. A* mixed Hodge structure *on $H$ is a tripel $(H, W, F)$ where $W$ is an increasing fitration on $H$ and $F$ is a decreasing filtration on $H_{\mathbb{C}}$, such that $F$ induces a pure Hodge structure of weight $k$ on $\mathrm{Gr}_k^W := W_k/W_{k-1}$. The filtration $W$ is called the* weight filtration*, and the filtration $F$ is called the* Hodge filtration*. Morphisms of mixed Hodge structures are $\mathbb{Q}$-linear maps preserving both filtrations. We denote the category of mixed Hodge structures by $\mathfrak{mhs}$.*

We have already seen that $\mathfrak{mhs}$ is abelian, and that $\mathrm{K}_0(\mathfrak{mhs}) = \mathrm{K}_0(\mathfrak{hs})$ holds canonically. At the end of the previous talk, we established:

*Theorem 1.1b ([Pet10, Thm.5.2.6.]) — Let $U$ be a complex algebraic variety, and $k \geqslant 0$ an integer. Then there exists a mixed Hodge structure on $\mathrm{H}^k(U)$.*

This is part of proving Deligne's theorem on mixed Hodge structures on cohomology of pairs (Thm.2.2.5. in [Pet10]).

With all this at hand, we can now state the first main theorem of this talk:

**Theorem 1.2.** *The map $\chi^{\mathrm{Hdg}}$ from (2) is given as*

$$\chi^{\mathrm{Hdg}}\colon\ \mathrm{K}_0(\mathrm{Var}) \to \mathrm{K}_0(\mathfrak{mhs}) = \mathrm{K}_0(\mathfrak{hs}),\ U \mapsto \sum_k [\mathrm{Gr}_k^W \mathrm{H}_c^k(U)],$$

*where $W$ is the weight filtration on $\mathrm{H}_c^k(U)$.*

We sketch the proof now, which relies on two main statements that are shown in the following two sections:

*Proof.* To avoid confusion, let's write $\widetilde{\chi}$ for the map $\widetilde{\chi}\colon U \mapsto [\mathrm{Gr}_k^W \mathrm{H}_c^k(U)]$ (defined on the level of varieties). We are going to show two things:

(i) The map $\widetilde{\chi}$ respects the scissor relations (1), and thus descends to a map $\widetilde{\chi}\colon\ \mathrm{K}_0(\mathrm{Var}) \to \mathrm{K}_0(\mathfrak{mhs})$.

(ii) The map $\widetilde{\chi}$ is a ring homomorphism.

Then the uniqueness of the Hodge characteristic, together with the fact that

$$\widetilde{\chi}|_{\mathrm{K}_0(\mathrm{Var}^{\mathrm{sm,proj}})} = \chi^{\mathrm{Hdg}}$$

implies the theoren (by Bittner's theorem). □

## 2 Scissor relations for the Hodge characteristic

### *The mixed cone*

**2.1.** Recall from previous talks: For any topological space $X$ and $\mathcal{F}$ a sheaf on $X$, we have the Godement resolution $C(\mathcal{F})$. If $f\colon X \to Y$ is a continous map and $\mathcal{G}^\bullet$ a complex of sheaf on $Y$, then $f$ induces a morphism of complexes

$$f^\sharp\colon \mathcal{G}^\bullet \to Rf_* f^{-1}\mathcal{G}^\bullet$$

Specializing to the constant local system, we have a map $f^\sharp\colon \underline{Y}_{\mathbb{Q}} \to Rf_*\underline{X}_{\mathbb{Q}}$ and isomorphisms

$$R\Gamma^q(Y, \mathrm{cone}(f^\sharp)) \xrightarrow{\sim} \widetilde{\mathrm{H}}^{q+1}(\mathrm{cone}(f), \mathbb{Q}),$$

where the right-hand side is the reduced cohomology of the topological space $\mathrm{cone}(f)$.

**Lemma 2.2.** *Let $i\colon V \hookrightarrow U$ be the inclusion of a closed set, with induced map $i^\sharp\colon \underline{U}_{\mathbb{Q}} \to i_*\underline{V}_{\mathbb{Q}}$. Then $\mathrm{H}^k(U, V) = R\Gamma^{k-1}(U, \mathrm{cone}(i^\sharp))$.*

**2.3.** Recall the somewhat lengthy definition of a mixed Hodge complex: It consists of a tupel $((K, W), (K_{\mathrm{C}}, W, F), \beta)$, where

- $(K, W)$ is a bounded below filtered complex of $\mathbb{Q}$-vector spaces,
- $(K_{\mathbb{C}}, W, F)$ is a bi-filtered complex of vector spaces, and

  $$\beta\colon (K, W) \to (K_{\mathbb{C}}, W) = \mathrm{Res}_{\mathbb{Q}\subseteq\mathbb{C}}((K_{\mathbb{C}}, W))$$

  is a “pseudo-morphism” in the category of bounded below filtered $\mathbb{Q}$-complexes[1], that moreover induces a “pseudo-isomorphism”

  $$\beta \otimes \mathrm{id}\colon (K, W) \otimes \mathbb{C} \xrightarrow{\sim} (K_{\mathbb{C}}, W)$$

  of bounded-below filtered $\mathbb{C}$-complexes;

these are required to satisfy that the triple

$$\mathrm{Gr}^W_m K := (\mathrm{Gr}^W_m K, (\mathrm{Gr}^W_m K_{\mathbb{C}}, F), \mathrm{Gr}^W_m \beta)$$

is a $\mathbb{Q}$-Hodge complex of weight $m$. We note that it is a non-trivial result that the cohomology groups of a mixed Hodge complex cary a mixed Hodge structure (this relies on Deligne’s “two filtration lemma”, c.f. [Del74, 7.2]).

**Definition 2.4.** Let $((K, W), (K_{\mathbb{C}}, W, F), \beta)$ and $((K', W'), (K'_{\mathbb{C}}, W', F'), \beta')$ be complex of mixed Hodge structures. A *morphism of mixed Hodge complexes* consists of the following data:

[1] In [Del74, (8.1.10)], the morphism $\beta$ is only required on the level of filtered $\mathbb{C}$-complexes

(i) A morphism $u\colon (K,W) \to (K',W')$ in $D^+F(A)$

(ii) A morphism $u_{\mathbb{C}}\colon (K_{\mathbb{C}},W,F) \to (K'_{\mathbb{C}},W',F')$ in $D^+F_2(\mathbb{C})$

subject to the coherence condition that the diagram

$$\begin{array}{ccc} K\otimes\mathbb{C} & \xrightarrow{u\otimes\mathrm{id}} & K'\otimes\mathbb{C} \\ {\scriptstyle\beta\otimes\mathrm{id}}\downarrow\wr & & \wr\downarrow{\scriptstyle\beta'\otimes\mathrm{id}} \\ K_{\mathbb{C}} & \xrightarrow[u_{\mathbb{C}}]{} & K'_{\mathbb{C}} \end{array} \tag{3}$$

commutes in $D^+F(\mathbb{C})$ (with respect to $W$). This leads to the category CMHS of complexes of mixed Hodge structures. Note that CMHS is not just the derived category of the category of chain complexes over the abelian category $\mathfrak{mhs}$!

**2.5.** That we require the commutativity of (3) only in $D^+F(\mathbb{C})$ is somewhat of a problem, because it means that on representatives, the diagram is only homotopy-commutative. This makes it difficult (or rather impossible) to define a good triangulated structure, and we already have this problem without any filtrations:[2] Consider the category $\mathcal{C}$ consisting of triples $(K,K_{\mathbb{C}},\beta)$, where $K\in D(\mathbb{Q})$, $K_{\mathbb{C}}\in D(\mathbb{C})$, and $\beta\colon K\to K_{\mathbb{C}}$ is a morphism in $D(\mathbb{Q})$ such that $\beta\otimes\mathrm{id}_{\mathbb{C}}\colon K\otimes\mathbb{C}\to K_{\mathbb{C}}$ is an isomorphism in $D(\mathbb{C})$; a morphism between $(K,K_{\mathbb{C}},\beta)$ and $(K',K'_{\mathbb{C}},\beta')$ are pairs $(u,u_{\mathbb{C}})$, where $u\colon K\to K'$ is a morphism in $D(\mathbb{Q})$, $u_{\mathbb{C}}\colon K_{\mathbb{C}}\to K'_{\mathbb{C}}$ is a morphism in $D(\mathbb{C})$ and the diagram

$$\begin{array}{ccc} K\otimes\mathbb{C} & \xrightarrow{u\otimes\mathrm{id}} & K'\otimes\mathbb{C} \\ {\scriptstyle\beta\otimes\mathrm{id}}\downarrow\wr & & \wr\downarrow{\scriptstyle\beta'\otimes\mathrm{id}} \\ K_{\mathbb{C}} & \xrightarrow[u_{\mathbb{C}}]{} & K'_{\mathbb{C}} \end{array}$$

is supposed to be commutative in $D(\mathbb{C})$, so again only up to an (unspecified) homotopy. To define the cone of $u$, we would like to consider the ordinary cones $(\mathrm{cone}(u),\mathrm{cone}(u_{\mathbb{C}}))\in D(\mathbb{Q})\times D(\mathbb{C})$ , and find a “canonical” comparison morphism $\gamma\colon \mathrm{cone}(u)\to\mathrm{cone}(u_{\mathbb{C}})$ that is compatible with $\beta$, $\beta'$ in the sense that the diagram

$$\begin{array}{ccccc} K\otimes\mathbb{C} & \xrightarrow{u\otimes\mathrm{id}} & K'\otimes\mathbb{C} & \longrightarrow & \mathrm{cone}(u)\otimes\mathbb{C} \\ {\scriptstyle\beta\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\beta'\otimes\mathrm{id}} & & \vdots{\scriptstyle\gamma\otimes\mathrm{id}} \\ K_{\mathbb{C}} & \xrightarrow[u_{\mathbb{C}}]{} & K'_{\mathbb{C}} & \longrightarrow & \mathrm{cone}(u_{\mathbb{C}}) \end{array}$$

commutes in $D(\mathbb{C})$. While it is possible to find such a morphism in $D(\mathbb{C})$ (which is an isomorphism by a form of the 3-by-3-lemma), it is only well-defined up to homotopy, and is unique only if the homotopy witnessing the commutativity of the left-hand side is fixed.

[2] I think this also explains problems with Defintion 3.2 in [Beĭ86].

**2.6.** The following construction is possible: Let $u\colon (K, W, F) \to (K', W', F')$ be a morphism of bifiltered complexes over some ring $R$. Then the *mixed cone* of $u$ is defined as the usual complex $\operatorname{cone}(u)$, i.e. the complex $K[1] \oplus K'$, with differentials

$$d^n := \begin{pmatrix} -d^{n+1} & 0 \\ -u^{n+1} & d^n \end{pmatrix} : K^{n+1} \oplus K'^{,n} \to K^{n+2} \oplus K'^{,n+1},$$

together with filtrations $W[1] \oplus W'$, $F \oplus F'$.

*Lemma 2.6a — Let* $(K, W)$ *be a complex of* $R$*-modules with an increasing filtration* $W$*, and let* $(K[1], W[1])$ *be the shifted complex with* $(W[1]_n K)[1]$*. Then*

$$\operatorname{Gr}^W_n(K[1], W[1]) = (\operatorname{Gr}^W_{n-1} K, W)[1].$$

*If* $(K, W, F)$ *is a complex of mixed Hodge structures over* $A$*, then*

$$\mathrm{H}^i(\operatorname{Gr}^W_n(K[1], W[1]), F) = \mathrm{H}^{i+1}(\operatorname{Gr}^W_{n-1} K, F)$$

*is a Hodge-structure of weight* $n + i$*.*

More precisely, we have

*Lemma 2.6b — Let*

$$\begin{array}{ccc} K & \xrightarrow{u} & K' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ L & \xrightarrow[v]{} & L' \end{array}$$

*be a diagram in* $\operatorname{Ch}(A)$ *for some ring* $A$*, and let* $h$ *be a homotopy making the diagram homotopy commutative. Then the morphism*

$$\gamma := \begin{pmatrix} \alpha & 0 \\ h & \alpha' \end{pmatrix} : \operatorname{cone}(u) \to \operatorname{cone}(v)$$

*renders the diagram*

$$\begin{array}{ccccc} K & \xrightarrow{u} & K' & \longrightarrow & \operatorname{cone}(u) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} & & \downarrow{\scriptstyle\gamma} \\ L & \xrightarrow[v]{} & L' & \longrightarrow & \operatorname{cone}(v) \end{array}$$

*homotopy-commutative. If* $\alpha, \alpha'$ *are (quasi-)isomorphisms, then so is* $\gamma$*.*

In particular, if $(K, W, F)$ and $(K', W', F')$ are complexes of mixed Hodge structures, then so is $\operatorname{cone}(u)$.

**2.7.** Both good compactifications and semi-simplicial hyperresolutions are functorial in the sense that any morphism $f\colon U \to V$ of complex varieties can be completed to a diagram

$$\begin{array}{ccccc} U & \hookrightarrow & Y & \xleftarrow{\varepsilon_Y} & Y_\bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle \overline{f}_\bullet} \\ V & \hookrightarrow & Z & \xleftarrow{\varepsilon_Z} & Z_\bullet \end{array},$$

where $U \hookrightarrow Y$ and $V \hookrightarrow Z$ are smooth compactifications, and $Y_\bullet \to Y$, $Z_\bullet \to Z$ are semi-simplicial resolutions (c.f. [PS08, Cor.5.9]). Note that this is not the same as claiming that resolutions of singularities are functorial, which is in general false for non-smooth morphisms (c.f. [Kol05, Ex.3.2]).

Now the cruicial thing to note is that the "Hodge complex" is preserved under pullback, i.e. that we have

$$\overline{f}_\bullet^* \,\mathcal{H}\mathrm{dg}^\bullet(Z_\bullet \log E_\bullet) \cong \mathcal{H}\mathrm{dg}^\bullet(Y_\bullet \log D_\bullet)$$

for $U \hookrightarrow V$ the inclusion of a closed subset. Now we can use that logarithmic Hodge complexes are complexes of free modules, so in particular, we can calculate their pushforward without having to derive further, and that the $(-)_* \vdash (-)^*$-adjunction extends to bifiltered complexes, so we get a natural map of complexes of sheaves of mixed Hodge structures

$$f^\sharp\colon \varepsilon_{Z,*}\,\mathcal{H}\mathrm{dg}^\bullet(Z_\bullet \log E_\bullet) \longrightarrow \overline{f}_*\varepsilon_{Y,*}\,\mathcal{H}\mathrm{dg}^\bullet(Y_\bullet \log D_\bullet),$$

which can be further rigidified in the sense that the all comparison isomorphism are not only quasi-isomorphisms but admit a homotopy-inverse (since this is possible for all quasi-isomorphisms of bounded $\mathbb{Q}$-complexes).[3] We can then view this morphism as a morphism in a triangulated subcategory of $DF(\mathbb{Q}) \times DF_2(\mathbb{C})$, which is triangulated via the cone from 2.6. The "natural" cohomology functors on this triangulated category are indeed compatible with the Hodge-structures (and also induce morphisms of Hodge-structures), so the distinguished triangle

$$\varepsilon_{Z,*}\,\mathcal{H}\mathrm{dg}^\bullet(Z_\bullet \log E_\bullet) \xrightarrow{f^\sharp} \overline{f}_*\varepsilon_{Y,*}\,\mathcal{H}\mathrm{dg}^\bullet(Y_\bullet \log D_\bullet) \longrightarrow \mathrm{cone}(f^\sharp)$$

[3] This is a delicate issue, that I don't feel super confident about: Already for $j\colon U \hookrightarrow X$ the inclusion of a smooth variety $U$ into a smooth-projective variety $X$, we have a "zig-zag" of comparison quasi-isomorphisms

$$(Rj_*\mathbb{Q}_U, \tau) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} (Rj_*\mathbb{C}_U, \tau) \xrightarrow{\sim} (Rj_*\Omega_U^\bullet, \tau) \xleftarrow{\sim} (j_*\Omega_U^\bullet, \tau) \xleftarrow{\sim} (\Omega_X^\bullet(\log D), \tau) \xrightarrow{\sim} (\Omega_X^\bullet(\log D), W),$$

where $W$ is the weight filtration on $\Omega_X^\bullet \log(D)$ and $D \coloneqq X \setminus U$ is a normal crossing divisor. So in particular, a "canonical" comparison isomorphism

$$(Rj_*\mathbb{Q}_U, \tau) \xrightarrow{\sim} (\Omega_X^\bullet(\log D), W)$$

only exists in $D^+F(\mathbb{C})$. However, since we are considering complexes of free modules over a field, we can always find (non-canonical?) homotopy-inverses, which should be enough (since we are only interested in cohomology in the end).

yields to a long-exact sequence of mixed Hodge-structures.

**Proposition 2.8.** *Let $X$ be an algebraic variety, and $Y \subseteq X$ a closed subvariety. Then*

$$\chi^{\mathrm{Hdg}}(X) = \chi^{\mathrm{Hdg}}(Y) + \chi^{\mathrm{Hdg}}(X \setminus Y).$$

*Proof.* Let $\overline{X}$ be a compactification of $X$, with $T = \overline{X} \setminus X$, and $\overline{Y} \subseteq \overline{X}$ the closure of $Y$. Then $\mathrm{H}_c^k(X \setminus Y) = \mathrm{H}^k(\overline{X}, \overline{Y} \cap T)$, and the associated long-exact sequence of mixed Hodge-structures reads as

$$\ldots \longrightarrow \mathrm{H}_c^k(X \setminus Y) \longrightarrow \mathrm{H}_c^k(X) \longrightarrow \mathrm{H}_c^k(Y) \longrightarrow \mathrm{H}_c^{k+1}(X \setminus Y) \longrightarrow \ldots$$

□

**Remark 2.9.** In [ElZe83], El Zein gives a different construction of the mixed Hodge structures on the cohomology of varieties. I didn't have time to read this paper carefully, but there's also a discussion of the (non-)functoriality of the cone of complexes of mixed Hodge structures and their triangulated structure, in particular in 3.1.

## 3 Künneth formula for mixed Hodge structures

**Proposition 3.1.** *Let $U, V$ be complex algebraic varieties. Then the topological Künneth isomorphism*

$$\bigoplus_{p+q=k} \mathrm{H}^p(U, \mathbb{Q}) \otimes \mathrm{H}^q(V, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^k(U \times V, \mathbb{Q})$$

*is an isomorphism of mixed Hodge structures.*

By strictness of morphism of mixed Hodge structures[4], it suffices to see that the Künneth map is a morphism of Hodge structures.

### *Recollection from previous talks*

**3.2.** Recall the notion of the *cube category* $\square$: It has objects given by finite subsets of $\mathbb{N}$, and for $I, J \subseteq \mathbb{N}$, we have

$$\mathrm{Hom}_{\square}(I, J) = \begin{cases} \{*\}, & \text{if } I \subseteq J; \\ \emptyset, & \text{otherwise.} \end{cases}$$

[4]In other words, the forgetful functor $\mathfrak{mhs} \to \mathrm{Vec}_{\mathbb{Q}}$ is conservative

More important for us is the $n$-truncated cube category $\square_n$ for some $n \geqslant 1$. It is the category induced by the lattice of subsets of $\{0, \ldots, n-1\}$. So for example, this is $\square_2$:

$$\begin{array}{ccccc} & & \{0\} & & \\ & \nearrow & & \searrow & \\ \emptyset & & & & \{0,1\} \\ & \searrow & & \nearrow & \\ & & \{1\} & & \end{array}$$

Now let $\mathcal{C}$ be any category. A *$n$-truncated cubical object* in $\mathcal{C}$ is a functor $\square_n^{\mathrm{op}} \to \mathcal{C}$. We denote the category of $n$-truncated cubical objects of $\mathcal{C}$ by $\mathrm{Cube}_n(\mathcal{C})$. Similarly, for $A \subseteq \mathbb{N}$ any finite subset of $\mathbb{N}$, we have the notion of the $A$-cubical category $\square_A$, and the notion of $A$-cubical objects in $\mathcal{C}$, which we denote by $\mathrm{Cube}_A(\mathcal{C})$. We saw last time that we can define, for any $\mathcal{C}$ with finite coproducts, a functor

$$\mathrm{Cube}_{n+1}(\mathcal{C}) \to \mathrm{SeSimp}_n^+(\mathcal{C}),$$

where $\mathrm{SeSimp}_n^+(\mathcal{C})$ denotes the category of $n$-truncated augmented semi-simplicial objects in $\mathcal{C}$.[5] This extends to a functor $\mathrm{Cube}(\mathcal{C}) \to \mathrm{SeSimp}^+(\mathcal{C})$ for untruncated objects.

**3.3.** We then introduced the notion of a semi-simplicial resolution:

*Definition 3.3a — Let $\epsilon \colon X_\bullet \to X$ be an augmented semi-simplicial complex. We say that $\epsilon$ is of* cohomological descent *if the natural map*

$$\epsilon^\sharp \colon \underline{\mathbb{Q}}_X \longrightarrow \underline{\mathbb{Q}}_{X_\bullet}$$

*is an isomorphism.*[6]

*Definition 3.3b — Let $X$ be a variety and $D \subseteq X$ a closed subvariety. A* semi-simplicial resolution *of $X$ is an augmented semi-simplicial variety $\epsilon \colon X_\bullet \to X$, such that all the maps $X_k \to X$ are proper, $X_k$ is smooth for all $k$, $\epsilon$ is of cohomological descent and the inverse image of $D$ on each irreducible component $X_k^i$ is either all of $X_k^i$, empty, or a divisor with simple normal crossings on $X_k^i$.*

*Definition 3.3c — Let $X_I$ be a cubical variety. We say that it is a* cubical hyperresolution *if its associated semi-simplicial variety is a semi-simplicial resolution.*

*Theorem 3.3d — Let $X$ be a variety of dimension $n$, and $T \subseteq X$ a Zariski closed subset with dense complement. Then there exists a $(n+1)$-truncated cubical hyperresolution $(X_I)$ of $(X, T)$.*

[5] These are functors $\triangle_n^{+,\mathrm{op}}$, where $\triangle_n$ is the $n$-truncated semi-simplicial category, i.e. the wide (containing all objects) suncategory of $\Delta_n$ containing only the monorphisms, and the $(-)^+$ stands for augmentation, i.e, we have an additional object $\emptyset \in \triangle_n^+$ with a unique morphism $\emptyset \to [0]$.

[6] Two documents on this topic that look helpful, but that I couldn't read before the talk, are these expository notes of Illusie [Ill09] and these more technical notes by B. Conrad [Con].

We have the following crterion for an augmentation map to be of cohomological descent:

*Proposition 3.3e — Let $\epsilon\colon X_\bullet \to X$ be an augmented semi-simplicial complex variety. If the geometric realization $|\epsilon|\colon |X_\bullet| \to |X|$ is proper and has contractible fibers, then $\epsilon$ is of cohomological descent.*

**TODO 1.** Can the cubical hyperresolution of a variety $X$ always be choosen in such a way that the fibers in the geometric realization of the associated semi-simplicial variety are contractible?

### *Compatibility with products*

**TODO 2.** In the book ([PS08, p.134f]), they use some vague barycentric subdivision argument to argue that for varieties $X$ and $Y$ with semi-simplicial resolutions $X_\bullet \to X$ and $Y_\bullet \to Y$ the relation $|X_\bullet \times Y_\bullet| \cong |X_\bullet| \times |Y_\bullet|$ holds. They then wan to use this to see that $X_\bullet \times Y_\bullet \to X \times Y$ is still of cohomological descent. However, they don't say why this should follow. My guess is that they want to use the criterion Proposition *3.3e*, but this would need that every semi-simplicial resolution has contractible fibers. I am not sure if this holds. But I also don't think that this is necessary: That the geometric realization functor commmutes with finite products is true as long as the target category is suitably chosen (we need a convinient category of topological spaces). This is the main reason why I outlined the definitions from previous talks, because I could not get my head around what happens in [PS08] at all.

**Definition 3.4.** Let $X_\bullet$ be an augmentend semi-simplicial variety. We define its derived category as

$$D(X_\bullet, \mathbb{Q}) := \lim_{\Delta^{\mathrm{inj}}} D(X_n, \mathbb{Q}),$$

where the limit is taken over the pullback maps $s_n^*\colon D(X_n, \mathbb{Q}) \to D(X_{n+1}, \mathbb{Q})$

We don't have to worry about any hyper-completeness issues here, as everything is (locally) compact.

**Proposition 3.5.** *Let $\epsilon_X : X_\bullet \to X$ and $\epsilon_Y : Y_\bullet \to Y$ be augemented semi-simplicial varieties over $\mathbb{C}$. Then the following diagram of stable $\mathbb{Q}$-linear $\infty$-categories commutes:*

$$\begin{array}{ccc} D(X_\bullet \times Y_\bullet, \mathbb{Q}) & \xleftarrow{\boxtimes_\bullet} & D(X_\bullet, \mathbb{Q}) \otimes_{\mathbb{Q}} D(Y_\bullet, \mathbb{Q}) \\ \downarrow{\scriptstyle R(\epsilon_X \times \epsilon_Y)_!} & & \downarrow{\scriptstyle R\epsilon_{X,!} \otimes R\epsilon_{Y,!}} \\ D(X \times Y, \mathbb{Q}) & \xleftarrow[\sim]{\boxtimes} & D(X, \mathbb{Q}) \otimes_{\mathbb{Q}} D(Y, \mathbb{Q}). \end{array}$$

Note that this is a statement purely on the derived category of $\mathbb{Q}$-modules on the schemes, and does not involve any statement about Hodge structures.

*Proof.* The lower isomorphism is [Vol23, 1.2.30] (note that varieties are locally compact). Now the point is that we can use the limit-description of $D(X_\bullet, \mathbb{Q})$, together with the fact that $R(-)_!$ can be computed termwise, to get the result from [Vol23, Prop.1.6.11]. □

**Corollary 3.6.** Let $X$ and $Y$ be varieties over $\mathbb{C}$, with semi-simplicial resolutions $\epsilon_X \colon X_\bullet \to X$ and $\epsilon_Y \colon Y_\bullet \to Y$. If $\epsilon_X$ and $\epsilon_Y$ are of cohomological descent, then so is $\epsilon_X \times \epsilon_Y$.

*Proof.* This now follows from the above proposition, together with the fact that since semi-simplicial resolutions leads to (degree-wise) proper maps, so $R(-)_!$ and $R(-)_*$ agree. Note that it is not necessary for the upper-horizontal map to be an isomorphism that this corollary holds, as we have:

$$\begin{aligned} R(\epsilon_X \times \epsilon_Y)_! \mathbb{Q}_{X_\bullet \times Y_\bullet} &\cong R(\epsilon_X \times \epsilon_Y)_! (\mathbb{Q}_{X_\bullet} \boxtimes_\bullet \mathbb{Q}_{Y_\bullet}) \\ &\cong R\epsilon_{X,!}\mathbb{Q} \boxtimes R\epsilon_{Y,!}\mathbb{Q} \\ &\cong \mathbb{Q}_X \boxtimes \mathbb{Q}_Y \\ &\cong \mathbb{Q}_{X \times Y}. \end{aligned}$$

□

Proposition 3.5 should also hold for (bi-)filtered complexes, since we can apply $\mathrm{Fun}(\mathbb{Z}, -)$ respectively $\mathrm{Fun}(\mathbb{Z} \times \mathbb{Z}^{\mathrm{op}}, -)$ to it.[7]

## 4 Some examples and more general theory

### *Mayer-Vietoris sequence associated to a resolution of singularities*

This is taken from [CMP17, pg.68-70] (see also [PS08, Thm.5.35]), but the proof is apparently not well-documented ([Cor16, pg.14]).

[7] In [Del74, (8.1.24)], it is just stated without any elaboration that for $j' \colon U' \hookrightarrow X'$ the inclusion of an open subset $U'$ into a smooth-projective $X'$ with complement a normal crossing divisor, and $j'' \colon U'' \to X''$ analogusly, the inclusion $j \colon U := U' \times U'' \hookrightarrow X' \times X'' =: X$ yields a quasi-isomorphism

$$Rj'_*\mathbb{Q} \boxtimes Rj''_*\mathbb{Q} \xrightarrow{\sim} Rj_*\mathbb{Q},$$

a filtered quasi-isomorphism

$$(Rj'_*\mathbb{Q}, \tau) \boxtimes (Rj''_*\mathbb{Q}, \tau) \xrightarrow{\sim} (Rj_*\mathbb{Q}, \tau)$$

and a bi-filtered morphism

$$\Omega^\bullet_{X'}(\log Y'), W, F) \boxtimes \Omega^\bullet_{X''}(\log Y''), W, F) \xrightarrow{\sim} \Omega^\bullet_X(\log Y), W, F).$$

But the first isomorphism already is the ordinary Künneth, as in [Vol23, Prop.1.6.11].

**Proposition 4.1.** *Let $X$ be a singular projective variety with singular locus $\Sigma$, and consider the resolution of singularities:*

$$\begin{array}{ccc} \widetilde{\Sigma} & \hookrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \Sigma & \underset{i}{\hookrightarrow} & X \end{array},$$

*where $\widetilde{\Sigma}$ is the exceptional locus. Then there is an associated long-exact sequence of mixed Hodge structures*

$$\ldots \to \mathrm{H}^m(\widetilde{\Sigma}, \mathbb{Q}) \to \mathrm{H}^m(X, \mathbb{Q}) \to \mathrm{H}^m(\widetilde{X}, \mathbb{Q}) \oplus \mathrm{H}^m(\Sigma, \mathbb{Q}) \to \mathrm{H}^{m-1}(\widetilde{\Sigma}, \mathbb{Q}) \to \ldots.$$

To quote [Cor16] — the cohomology of $X$ is sliced by putting a ration weight filtration on it the graded pieces of which are either pure Hodge structures, or mixed Hodge structures of lower-dimensional schemes.

*Idea of proof, as in [PS08].* This statement holds for any 2-cubical variety which is of cohomological descent. The idea is to cover a 2-cubical variety

$$\begin{array}{ccc} U & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

by the diagram of cubical hyperresolutions

$$\begin{array}{ccc} U_\bullet & \longrightarrow & Z_\bullet \\ \downarrow & & \downarrow \\ Y_\bullet & \longrightarrow & X_\bullet \end{array}$$

which induces a quasi-isomorphism[8]

$$\mathcal{H}\mathrm{dg}^\bullet(X_\bullet) \xrightarrow{\sim} \mathrm{cone}^\bullet(\mathcal{H}\mathrm{dg}^\bullet(Y_\bullet) \oplus \mathcal{H}\mathrm{dg}^\bullet(Z_\bullet) \to \mathcal{H}\mathrm{dg}^\bullet(U_\bullet))[-1].$$

So we have a distinguished triangle of the form

$$\mathcal{H}\mathrm{dg}^\bullet(X_\bullet) \to \mathcal{H}\mathrm{dg}^\bullet(Y_\bullet) \oplus \mathcal{H}\mathrm{dg}^\bullet(Z_\bullet) \to \mathcal{H}\mathrm{dg}^\bullet(U_\bullet),$$

and the long-exact cohomology sequence of this triangle is what we want. □

More examples on this can be found in [Cir21].

[8] again, this should hold on the level of topological spaces, but I am not sure why it respects the Hodge-structure

## 5 The Leray spectral sequence is motivic

**5.1.** Let $f\colon Y \to X$ be a morphism of algebraic varieties, and the induced derived pushforward $Rf_*\colon D(Y,\mathbb{Q}) \to D(X,\mathbb{Q})$. Associated to it we have the Leray spectral sequence (which is a special instance of the Grothendieck spectral sequence for the composition with the structure maps to $\mathrm{Spec}(\mathbb{C})$),

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i(X, \mathrm{H}^j(Rf_*\mathcal{F})) \Longrightarrow \mathrm{H}^{i+j}(Y,\mathcal{F})$$

for any complex $\mathcal{F}$ of local systems on $Y$. The goal now is to show that this is a spectral sequence of mixed Hodge structures. This is due to Arapura ([Ara04]). We only mention this in passing, because it also goes under the name motivic. In the publication, they use this approach to calculate the motivic structure on $\mathrm{H}^3(X)$ for a smooth projective treefold over $\mathbb{C}$ that admits a flat map $X \to S$ that has fibers given by connected rational curves, but I do not understand enough geometry to elaborate.

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