Galois gerbes and $\mathcal{E}^{\mathrm{rig}}$

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We outline the construction of Kaletha's Galois gerbe $\mathcal{E}^{\mathrm{rig}}$ that is used for the parametrization of rigid inner forms, following [Kal16]. These notes were prepared for the gradutate seminar on the local Langlands conjecturs for non quasi-split groups, which took place in Bonn during the summer semester 2023. I would like to thank Zhen Huang, Han Jiadong, and David Schwein for helping me prepare for this talk. Some claims and questions I could not resolved are marked in pink, additions for clarification after the talk in blue.

Contents

1 Recollections **1**
2 Galois gerbes in characteristic 0 **3**
3 Construction of $\mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G)$ **6**
4 Extending the Kottwitz isomorphism **11**
References **12**

Throughout, F denotes a local field of characteristic zero.

1 Recollections

1.1. One of the main features of Galois cohomology is Tate-Nakayama duality, which can be understood as an isomorphism

$$\mathrm{H}^1(\Gamma, T) = \hat{\mathrm{H}}^1(F, T) \xrightarrow{\sim} \hat{\mathrm{H}}^{-1}(E/F, \mathbb{X}_\bullet(T)) = \mathbb{X}_\bullet(T)_\Gamma[\mathrm{tor}],$$

where T is a torus, E/F a finite Galois extension that splits T, and $\mathbb{X}_\bullet(T)_\Gamma[\mathrm{tor}]$ is the torsion part of the Galois coinvariants of the action on the cocharacters $\mathbb{X}_\bullet(T)$. Kottwitz gives an interpretation of the right-hand side in terms of the dual group, and obtains an isomorphism[1]

$$\mathrm{H}^1(\Gamma, T) \xrightarrow{\sim} \pi_0(\widehat{T}^\Gamma)^D.$$

This morphism in fact extends to all reductive groups:

Theorem 1.1a ([Kot86, Thm.1.2]) — Let G be a connected reductive group over a local field F of characteristic zero. Then there is a unique morphism

$$\alpha\colon\ \mathrm{H}^1(\Gamma, G) \longrightarrow \pi_0(Z(\hat{G}))^D,$$

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[1] An expository account of this can be found in [Dri].

that extends the Tate-Nakayama isomorphism, in the sense that for any maximal torus T of G, the diagram

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma, T) & \longrightarrow & \mathrm{H}^1(\Gamma, G) \\ \downarrow & & \downarrow \\ \pi_0(\hat{T}^\Gamma)^D & \longrightarrow & \pi_0(Z(\hat{G})^\Gamma)^D \end{array}$$

commutes. If F is p-adic, then this is an isomorphism. If $F = \mathbb{R}$, then the kernel and image can be explicitly described.

1.2. This is helpful because we can use it to calculate the Galois cohomology groups that classify different variants of "inner twists": Recall that for a reductive algebraic group G, we have the bijection

$$\begin{array}{c} \{\text{inner twists } \xi\colon G \to G'\}/_{\text{iso. of inner twists}} \\ \downarrow\wr \\ \mathrm{H}^1(\Gamma, G_{\mathrm{adj}}). \end{array}$$

We also saw that we have

$$\begin{array}{c} \{\text{pure inner twists } (\xi\colon G \to G', z)\}/_{\text{iso. of pure inner twists}} \\ \downarrow\wr \\ \mathrm{H}^1(\Gamma, G), \end{array}$$

and that these notions are compatible in the sense that the diagram

$$\begin{array}{ccc} \{\text{pure inner twists of } G\}/_\sim & \xrightarrow{(\xi, G', z) \mapsto (\xi, G')} & \{\text{inner twists of } G\}/_\sim \\ \wr\downarrow & & \downarrow\wr \\ \mathrm{H}^1(\Gamma, G) & \longrightarrow & \mathrm{H}^1(\Gamma, G_{\mathrm{adj}}) \end{array}$$

commutes. However, the lower horizontal arrow is not necessarily surjective or injective:

Example 1.3. Consider the case of $G = \mathrm{SL}_n$ over $\mathbb{Q}_p$. We have an explicit description of the inner forms of G via central simple F-algebras of F-dimension n^2 every inner form of $\mathrm{SL}_n(\mathbb{Q}_p)$ is of the form $\mathrm{GL}_m(D)_{\mathrm{der}}$, where D is is a division algebra over F of dimension d^2 and $n = md$ holds. Now the dual group of $\mathrm{SL}_n(F)_{\mathrm{adj}}$ is given by $\widehat{\mathrm{SL}_n(\mathbb{Q}_p)}_{\mathrm{adj}} = \mathrm{SL}_n(\mathbb{C})$.[2] We can then also use the Kottwitz isomorphism to calculate both $\mathrm{H}^1(\Gamma, G)$ and $\mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$:

[2] We know that taking $\widehat{(-)}$ interchanges being of adjoint and of simply-connected type, and we have $\widehat{\mathrm{SL}_n(\mathbb{Q}_p)} = \mathrm{PGL}_n(\mathbb{C}) = \mathrm{PSL}_n(\mathbb{C})$ (since $\mathbb{C}$ is algebraically closed), which clearly admits the simply-connected cover $\mathrm{SL}_n(\mathbb{C})$.

- For G, we have $\mathrm{H}^1(\Gamma, G) = \pi_0(Z(\widehat{G})^\Gamma)^D = \pi_0(Z(\mathrm{PGL}_n(\mathbb{C}))^\Gamma)^D = \{1\}$, since $\mathrm{PGL}_n(\mathbb{C})$ has trivial center. *We see that the only pure inner twist of* SL_n *is the trivial pure inner twist!*
- For G_{adj}, we have

$$\mathrm{H}^1(\Gamma, G_{\mathrm{adj}}) = \pi_0(Z(\widehat{G_{\mathrm{adj}}})^\Gamma)^D = \pi_0(Z(\mathrm{SL}_n(\mathbb{C}))^\Gamma)^D = (\mu_n)^D,$$

  since the Galois action is trivial.

In fact, we always encounter this problem when G is p-adic and simply connected, as then $\mathrm{H}^1(\Gamma, G) = \{1\}$.

Example 1.4. Consider the case of U_n^* over $\mathbb{Q}_p$ for n odd. Then the natural map $\mathrm{H}^1(\Gamma, \mathrm{U}_n^*) \to \mathrm{H}^1(\Gamma, \mathrm{U}_{n,\mathrm{adj}}^*)$ is given by the projection $\mathbb{Z}/2\mathbb{Z} \to \{1\}$, i.e. U_n^* does not have a non-trivial inner form, but there are two non-equivalent ways to view U_n^* as an pure inner form of itself.

Proof. We claim that in the case n odd, we have

$$\mathrm{H}^1(\Gamma, \mathrm{U}_n^*) = \mathbb{Z}/2\mathbb{Z} \text{ and } \mathrm{H}^1(\Gamma, \mathrm{U}_{n,\mathrm{adj}}^*) = \{1\}.$$

Again, we can use the Kottwitz isomorphism

$$\mathrm{H}^1(\Gamma, G) = \pi_0(Z(\hat{G})^\Gamma)^D$$

in both cases (where $\hat{G}$ is the Langlands dual group). Now, we have $\widehat{\mathrm{U}_n^*} = \mathrm{GL}_n(\mathbb{C})$, and the Galois action factors over $\Gamma_{E/F}$, where the non-trivial element σ acts via $g \mapsto \mathrm{Adj}(J_n)g^{-1,t}$, for

$$J_n = \begin{bmatrix} & & & & -1 \\ & & & 1 & \\ & & -1 & & \\ & \iddots & & & \\ (-1)^n & & & & \end{bmatrix}$$

Moreover, we have $\mathrm{U}_{n,\mathrm{adj}}^* = \mathrm{U}_n^*/\mathrm{U}_1^*$, and $\widehat{\mathrm{U}_{n,\mathrm{adj}}^*} = \mathrm{SL}_n(\mathbb{C})$, with the same action.

Now the restriction of the Galois action to $Z(\mathrm{GL}_n(\mathbb{C})) = \mathbb{C}^\times$ and $Z(\mathrm{SL}_n(\mathbb{C})) = \mu_n$ is in both cases given by complex conjugation. So $\mathrm{H}^1(\Gamma, \mathrm{U}_n^*) = \pi_0(\mathbb{C}^{\times,(-)^{-1}})^D \cong \mathbb{Z}/2\mathbb{Z}$ (since we can restrict to the action on S^1, where it is given by complex conjugation), and $\mathrm{H}^1(\Gamma, \mathrm{U}_{n,\mathrm{adj}}^*) \cong \{1\}$ (since n is odd). □

2 Galois gerbes in characteristic 0

Definition 2.1 ([LR87],[Kot14]). Assume $\mathrm{char}(F) = 0$. A *Galois gerbe* is a group extension

$$1 \longrightarrow u(\bar{F}) \longrightarrow \mathcal{E} \longrightarrow \Gamma \longrightarrow 1,$$

where $u(\bar{F})$ is the $\bar{F}$-valued points of an abelian F-group scheme u, and such that the action of Γ on $u(\bar{F})$ is given as

$$\sigma \cdot a = \hat{\sigma} a \hat{\sigma}^{-1}$$

with $\hat{\sigma} \in \mathcal{E}$ a preimage of σ. We call the group $u(\bar{F})$ the *band* of the gerbe. Two such extensions are called equivalent if there is a commutative diagram of the form

$$\begin{array}{ccccccccc} 1 & \longrightarrow & u(\bar{F}) & \longrightarrow & \mathcal{E}' & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle\sim} & & \| & & \\ 1 & \longrightarrow & u(\bar{F}) & \longrightarrow & \mathcal{E} & \longrightarrow & \Gamma & \longrightarrow & 1 \end{array}$$

Definition 2.2. Let G be an algebraic group defined over F, and $Z \to G$ with Z a finite multiplicative group whose image is contained in the center of G. Let $u \to \mathcal{E}$ be a Galois gerbe. The set $G(\bar{F})$ carries a continous Γ-action, which can be inflated to a continous $\mathcal{E}$-action.[3] Let

$$\mathrm{Z}^1(u \to \mathcal{E}, Z \to G) \subseteq \mathrm{Z}^1_{\mathrm{cont}}(\mathcal{E}, G(\bar{F}))$$

consist of these continous cocycles $f \colon \mathcal{E} \to G(\bar{F})$ such that their restriction to $u(\bar{F}) \subseteq \mathcal{E}^{\mathrm{rig}}$ factors as

$$\begin{array}{ccc} u(\bar{F}) & \xrightarrow{\widetilde{f}(\bar{F})} & Z(\bar{F}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{E} & \xrightarrow[f]{} & G(\bar{F}) \end{array}$$

where $\widetilde{f}(\bar{F})$ is the map on $\bar{F}$-valued points induced by a morphism of group schemes $\bar{f} \colon u \to Z$. Now $\mathrm{Z}(u \to \mathcal{E}, Z \to G)$ is closed under the equivalence relation of being a 1-coboundary, and so we can define

$$\mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G) := \mathrm{Z}(u \to \mathcal{E}, Z \to G) / \mathrm{B}(\mathcal{E}, G(\bar{F})),$$

which is naturally a pointed set.

Lemma 2.3. *Let G be an algebraic group, and $u \to \mathcal{E}$ a Galois gerbe. Then the canonical map $\mathrm{H}^1(\Gamma, G) \to \mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$ factors as*

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma, G) & \longrightarrow & \mathrm{H}^1(\Gamma, G_{\mathrm{adj}}) \\ & \searrow \quad \nearrow & \\ & \mathrm{H}^1(u \to \mathcal{E}, Z \to G) & \end{array}$$

for any finite central subgroup $Z \to G$.

[3] i.e. $\gamma \in \mathcal{E}$ acts by via its image in Γ

So the cohomology set $\mathrm{H}^1(u \to \mathcal{E}, Z \to G)$ is a good candidate for a set that is "big enough" for the map to $\mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$ to be surjective.

Proposition 2.4 ([NSW08, (2.7.7)])**.** *Then there is a bijection between Galois gerbes*

$$1 \longrightarrow u(\bar{F}) \longrightarrow \mathcal{E} \longrightarrow \Gamma \longrightarrow 1$$

and the set $\mathrm{H}^2(\Gamma, u)$*.*

Sketch of proof. Let us sketch the proof in the case that $u(\bar{F})$ is finite:

- Suppose we are given such an extension. The map $\mathcal{E} \to \Gamma$ admits a continous section $s\colon \Gamma \to \mathcal{E}$, which is not necessarily a group homomorphism (this is the case if and only if the extension splits), but we can still assume $s(1) = 1$. Consider now the map

$$\phi_s\colon \Gamma^2 \to \mathcal{E},\ (g_1, g_2) \mapsto s(g_1)s(g_2)s(g_1g_2)^{-1}.$$

  This map becomes the constant map after composing with the projection $\mathcal{E} \to \Gamma$, so factors over $u(\bar{F}) \subseteq \mathcal{E}$, i.e. we obtain a continous 2-cochain. One can then check that this map is in fact a 2-cocycle, and that for any other choice of section s' the associated map $\phi_{s'}$ is cohomologous to ϕ_s, so the extension gives a well-defined element of $\mathrm{H}^2(\Gamma, u)$. One then further checks that two equivalent extensions give rise to the same cohomology class.

- For the other direction, let $\phi\colon \Gamma \times \Gamma \to u(\bar{F})$ be a continous 2-cocycle. We can assume that ϕ is "normalized", i.e. that $\phi(1, g) = \phi(g, 1) = 1$ holds for all $g \in \Gamma$ (more precisely, the cohomology class of $\phi \in \mathrm{H}^2(u, \Gamma)$ contains a normalized representative). We then use ϕ to define a group structure on the topological space $u(\bar{F}) \times \Gamma$ (endowed with the product topology), via

$$(u_1, g_1) \star_\phi (u_2, g_2) \coloneqq (u_1 \cdot g_1u_2 \cdot \phi(g_1, g_2), g_1g_2),$$

  where we use the explicit description of the Γ-action on $u(\bar{F})$ to make sense of the factor g_1u_2. One then uses the cocycle property and that ϕ is normalized to calculate that this defines a group structure on $u(\bar{F}) \times \Gamma$. One then further verifies that the canonical inclusion and projection become group homomorphisms, and that any cohomologous normalized cocycle yields an equivalent extension.

□

Remark 2.5. We can also describe the group $\mathrm{H}^1(u, \Gamma)$ in this setup. Namely, for any extension

$$1 \to u(\bar{F}) \to \mathcal{E} \xrightarrow{\pi} \Gamma \to 1,$$

we can consider the group $\mathrm{Aut}([\mathcal{E}])$ of automorphisms of the extension. Say that two such automorphisms f_1, f_2 are equivalent if $f_1 = \gamma_u \circ f_2$ holds, where $\gamma_m\colon \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ is given by conjugation with some element in $u(\bar{F})$, and consider the quotient $\mathrm{Aut}([\mathcal{E}])/\sim$ by this equivalence relation. One can then check that for any continous cocycle $\phi\colon \Gamma \to u(\bar{F})$ and $f \in \mathrm{Aut}([\mathcal{E}])/\sim$, the morphism

$$\mathcal{E} \to \mathcal{E},\ x \mapsto \phi(\pi(x)) \cdot f(x)$$

is again an element of $\mathrm{Aut}([\mathcal{E}])$, and that this descends to a well-defined action of $\mathrm{H}^1(u, \Gamma)$ on $\mathrm{Aut}([\mathcal{E}])/\sim$, that is moreover simply transitive.

The idea now is to find such a u and a "distinguished" element in $\mathrm{H}^2(\Gamma, u)$ to parametrize inner forms. Note that in the trivial case $u = \{1\}$, we just recover ordinary Galois cohomology.

2.6. Before we continue, let us make the general observation that for any G algebraic, the long-exact cohomology sequence for $Z(G) \hookrightarrow G \twoheadrightarrow G_{\mathrm{adj}}$ gives rise to the long-exact sequence of pointed sets

$$\mathrm{H}^1(\Gamma, G) \longrightarrow \mathrm{H}^1(\Gamma, G_{\mathrm{adj}}) \xrightarrow{\delta} \mathrm{H}^2(\Gamma, Z(G)),$$

and so for a class $\varepsilon \in \mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$, to it lying in the image of $\mathrm{H}^1(\Gamma, G) \to \mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$ is given by the class $\delta(\varepsilon) \in \mathrm{H}^2(\Gamma, Z(G))$. The point now is that $\delta(\varepsilon)$ defines a Galois gerbe $\mathcal{E}^{\varepsilon}$ banded by $Z(G)$, and inflation is compatible with this in the sense that the diagram

$$\begin{array}{ccccc}
\mathrm{H}^1(\mathcal{E}^{\varepsilon}, G) & \longrightarrow & \mathrm{H}^1(\mathcal{E}^{\varepsilon}, G_{\mathrm{adj}}) & \longrightarrow & \mathrm{H}^2(\mathcal{E}^{\varepsilon}, Z(G)) \\
\uparrow{\scriptstyle \inf} & & \uparrow{\scriptstyle \inf} & & \uparrow{\scriptstyle \inf} \\
\mathrm{H}^1(\Gamma, G) & \longrightarrow & \mathrm{H}^1(\Gamma, G_{\mathrm{adj}}) & \longrightarrow & \mathrm{H}^2(\Gamma, Z(G))
\end{array}$$

commutes.[4] The point now is that $\inf(\delta(\varepsilon)) = 0$ holds in $\mathrm{H}^2(\mathcal{E}^{\varepsilon}, Z(G))$ ([Gir71, VIII.5]). In other words, passing from Γ to $\mathcal{E}^{\varepsilon}$ alows us to at least have ε in the image of the map $\mathrm{H}^1(\mathcal{E}^{\varepsilon}, G) \to \mathrm{H}^1(\mathcal{E}^{\varepsilon}, G_{\mathrm{adj}})$. In particular, if $\mathrm{H}^2(\Gamma, Z(G))$ is cyclic (eg. $G = \mathrm{SL}_n$), then passing to the gerbe defined by a generator already makes the map $\mathrm{H}^1(\mathcal{E}, G) \to \mathrm{H}^1(\mathcal{E}, G_{\mathrm{adj}})$ surjective.

3 Construction of $\mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G)$

Construction 3.1. Let F be a local field of characteristic zero, with fixed algebraic closure $\bar{F}$. Let $R_{E/F}[n] \coloneqq \mathrm{Res}_{E/F}\, \mu_n$ be the Weil restriction of scalars of the group μ_n of n-th roots of unity. Explicitly, if K/F is a Galois extension, we have

$$R_{E/F}[n](K) = \mathrm{Hom}(\Gamma_{E/F}, \mu_n(\bar{F}))^{\Gamma_K},$$

where the Galois action is given by "conjugation" in the sense of

$$(\sigma \cdot f)(\tau) \coloneqq \sigma(f(\sigma^{-1}\tau)),$$

[4] I did not check this for non-abelian cohomology, but that it holds in the abelian case is [NSW08, (1.5.2)].

for $f\colon \Gamma_{E/F} \to \mu_n(\bar{F})$, $\sigma \in \Gamma_K$ and $\tau \in \Gamma_{E/F}$. We have a natural "diagonal" inclusion $\mu_n \hookrightarrow R_{E/F}[n]$, given on K-valued points by mapping $x \in \mu_n(K)$ to the the constant map with value $x \in \mu_n(\bar{F})$ (as we have, by assumption, $\mu_n(K) \subseteq \mu_n(\bar{F})$). Let $u_{E/F,n}$ be the cokernel of this map, so that we have an exact sequence of commutative group schemes

$$1 \longrightarrow \mu_n \longrightarrow R_{E/F,n}[n] \longrightarrow u_{E/F,n} \longrightarrow 1$$

For every tower of Galois extension $F \subseteq K \subseteq E$ and $m \in \mathbb{N}$ a multiple of n, we have a natural map $p\colon R_{K/F}[m] \to R_{E/F}[n]$, which can be described on points as

$$(pf)(a) = \prod_{\substack{b \in \Gamma_{K/F} \\ b \mapsto a}} f(b)^{m/n}.$$

This is a surjection, and induces a surjection $u_{K/F,m} \twoheadrightarrow u_{E/F,n}$. Let

$$u := \varprojlim u_{E/F,n} \tag{1}$$

be the inverse limit over the index category $\mathcal{I}$ consisting of pair $(E/F, n)$ with E/F Galois, $n \in \mathbb{N}$, and

$$\mathrm{Maps}_{\mathcal{I}}((K/F, m), (E/F, n)) = \begin{cases} \{*\}, & \text{if } E \subseteq K \text{ and } n \mid m; \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that the $\bar{F}$-points of u have a natural profinite topology.[5] Moreover, $u(\bar{F})$ has a natural continous Γ-action, induced from the natural Γ-action on $R_{E/F,n}(\bar{F})$ by post-composition. Since u is abelian, we can thus talk of the continous cohomology groups $\mathrm{H}^i(\Gamma, u) := \mathrm{H}^i(\Gamma, u(\bar{F}))$.

Theorem 3.2. *We have*

$$\mathrm{H}^1(\Gamma, u) = 0 \text{ and } \mathrm{H}^2(\Gamma, u) = \begin{cases} \hat{\mathbb{Z}}, & \text{if } F \text{ is non-archimidean;} \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } F = \mathbb{R}. \end{cases}$$

Proof. For F any non-archimidean local field, we have in fact

$$\mathrm{H}^1(\Gamma, u) = \varprojlim_{E/F, n \geqslant 1} \mathrm{H}^1(\Gamma, \mathrm{Res}_{E/F} \mu_n) \text{ and } \mathrm{H}^2(\Gamma, u) = \varprojlim_{E/F, n \geqslant 1} \mathrm{H}^2(\Gamma, \mathrm{Res}_{E/F} \mu_n),$$

c.f. [Dil20, Prop.3.1]. Let us show why this holds and how this implies the claim in the case that F is p-adic (this is also the proof in [Far22, 5.1]): First, we have an exact sequence of the form [Sta, Tag 07KY]

$$0 \longrightarrow \mathrm{R}^1\!\lim_{E/F, n \geqslant 1} \mathrm{H}^1(\Gamma, \mathrm{Res}_{E/F} \mu_n) \longrightarrow \mathrm{H}^2(\Gamma, u) \longrightarrow \varprojlim_{E/F, n \geqslant 1} \mathrm{H}^1(\Gamma, \mathrm{Res}_{E/F} \mu_n) \longrightarrow 0$$

[5] Since we have $u(\bar{F}) = \varprojlim u_{E/F,n}(\bar{F})$, since taking global sections commutes with limits (it's a right-adjoint), and each of the $u_{E/F,n}(\bar{F})$ is endowed with the discrete topology

Now we have $\mathrm{H}^1(\Gamma, \mathrm{Res}_{E/F}\,\mu_n) = E^\times/E^{\times,n}$ by Kummer theory, which is finite, so in particular, the R^1lim-term above vanishes, and we have

$$\mathrm{H}^2(\Gamma, u) \xrightarrow{\sim} \varprojlim_{E/F, n \geqslant 1} \mathrm{H}^2(\Gamma, \mathrm{Res}_{E/F}\,\mu_n).$$

The groups on the right-hand side become acessible via class-field theory: we have

$$\mathrm{H}^2(\Gamma, \mathrm{Res}_{E/F}\,\mu_n) = \mathrm{Br}(E)[n] = \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

and the transition maps are given by multiplication, i.e. for (E', m) and (E, n) in the indexing category, the transition map is given by

$$\mathrm{Br}(E')[m] = \tfrac{1}{m}\mathbb{Z}/\mathbb{Z} \xrightarrow{\cdot m/n} \tfrac{1}{n}\mathbb{Z}/\mathbb{Z} = \mathrm{Br}(E)[n].$$

This gives $\mathrm{H}^2(\Gamma, u) = \hat{\mathbb{Z}}$ canonically, as desired.

□

Definition 3.3. Let $\xi = -1 \in \hat{\mathbb{Z}} = \mathrm{H}^2(\Gamma, u)$. Then this corresponds to a unique extension of profinite groups

$$1 \longrightarrow u(\bar{F}) \longrightarrow \mathcal{E}^{\mathrm{rig}} \longrightarrow \Gamma \longrightarrow 1\ . \tag{2}$$

We call $\mathcal{E}^{\mathrm{rig}}$ the *Kaletha gerbe.*[6]

Gainig a better understanding of $\mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G)$ is the main objective of the rest of this talk.

Basic properties of $\mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G)$

Lemma 3.4. *Let $Z \to G$ be as before, with G algebraic and Z finite central. The inflation-restriction sequence associated to (2) induces an inflation-restriction sequence for $\mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G)$,*[7] *i.e. the following diagram is commutative with exact rows:*

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{H}^1(\Gamma, G) & \xrightarrow{\mathrm{inf}} & \mathrm{H}^1_{\mathrm{cont}}(\mathcal{E}^{\mathrm{rig}}, G) & \xrightarrow{\mathrm{res}} & \mathrm{H}^1(u, G)^\Gamma & \xrightarrow{\mathrm{trg}} & \mathrm{H}^2(\Gamma, G) \\ & \| & & \uparrow & & \uparrow & & \| \\ 0 \longrightarrow & \mathrm{H}^1(\Gamma, G) & \longrightarrow & \mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G) & \longrightarrow & \mathrm{Hom}(u, Z)^\Gamma & \longrightarrow & \mathrm{H}^2(\Gamma, G) \end{array}$$

[6] In the original paper, the notation W is used instead of $\mathcal{E}^{\mathrm{rig}}$. We opted for $\mathcal{E}^{\mathrm{rig}}$ because it avoids confusion with the Weil group, and also because it seems to be better suited for the comparison with the gerbe $\mathcal{E}^{\mathrm{iso}}$ that parametrizes extended pure inner forms which we will see in the next talk.

[7] Recall that for a profinite group Γ with normal subgroup $N \subseteq G$ and A a G-module, there is always the inflation-restriction-sequence

$$0 \longrightarrow \mathrm{H}^1(\Gamma/N, A^N) \longrightarrow \mathrm{H}^1(\Gamma, A) \longrightarrow \mathrm{H}^1(N, A)^{\Gamma/N} \longrightarrow \mathrm{H}^2(\Gamma/N, A^N),$$

where we disregard the last term if A is not abelian. For A abelian, this is part of the five-term sequence associated to the Hochschild-Serre spectral sequence $\mathrm{E}_2^{i,j} = \mathrm{H}^i(\Gamma/N, \mathrm{H}^j(N, A)) \Rightarrow \mathrm{H}^{i+j}(\Gamma, A)$, which in turn is an instance of the Grothendieck spectral sequence associated to the factorization of the G-invariants as $(-)^G \colon G\text{-Mod} \xrightarrow{(-)^N} G/N\text{-Mod} \xrightarrow{(-)^{G/N}} \mathrm{Ab}$.

where we ignore $\mathrm{H}^2(\Gamma, G)$ *if* G *is not abelian.*

Note that we have the factorization over $\mathrm{Hom}(u, Z)^\Gamma$ since the restricted action of u is trivial. Moreover, since for any continous cocycle $z \in \mathrm{Z}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G)$, the image of $u(\bar{f})$ under z in G is contained in $Z(G) = \ker(G \twoheadrightarrow G_{\mathrm{adj}})$, we get a factorization in the following diagram

$$\begin{array}{ccccc} u(\bar{F}) & \hookrightarrow & \mathcal{E}^{\mathrm{rig}} & \twoheadrightarrow & \Gamma \\ {\scriptstyle z|_u}\downarrow & & \downarrow{\scriptstyle z} & & \downarrow \\ Z & \hookrightarrow & G & \twoheadrightarrow & G_{\mathrm{adj}} \end{array}$$

or, in other words, the image of z in $\mathrm{Z}^1(\mathcal{E}^{\mathrm{rig}}, G_{\mathrm{adj}})$ belong to $\mathrm{Z}(\Gamma, G_{\mathrm{adj}})$, and thus gives rise to an inner twist G^z of G.

Lemma 3.5. *The set* $\mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G)$ *is finite.*

Proof. We do this using the exact sequence from Lemma 3.4, which tells us that it is enough to see that $\mathrm{Hom}(u, Z)^\Gamma$ is finite and that the map $f\colon\ \mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G) \to \mathrm{Hom}(u, Z)^\Gamma$ has finite fiber over its image. For the first statement, we use the presentation

$$\mathrm{Hom}(u, Z)^\Gamma = \varinjlim \mathrm{Hom}(u_{E/F,n}, Z)^\Gamma$$

now the article says that since Z is finite, each of the terms appearing in the filtered colimit are finite (which i understand), but this doesn't necessarily imply that the colimit is again finite, right? For the second assertion, we use that for any $[z] \in \mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G)$ with lift $z \in \mathrm{Z}(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G)$, the fiber over $f(z)$ is given by $\mathrm{H}^1(\Gamma, G^z)$, where G^z is the inner form of G obtained from twisting by z (for more on twisting, see e.g. [Ser97, Prop.35bis] or [Knu+98, §28]).[8] It is then a general statement that $\mathrm{H}^1(\Gamma, G)$ is finite for any algebraic group G defined over the local field F (c.f. [PRR93, Thm.6.14]). □

Theorem 3.6. *Let* G *be a connected reducitve group over* F*, let* Z *be the center of* G_{der}*, and set* $\bar{G} := G/Z$*. Then there are natural surjections*

$$\mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G) \xrightarrow{a}\!\!\!\!\rightarrow \mathrm{H}^1(\Gamma, \bar{G}) \twoheadrightarrow \mathrm{H}^1(\Gamma, G_{\mathrm{adj}}).$$

Assume moreover that G *is split.*

[8] I think one can argue as follows: we know that the fiber of res over the distinguished element is given by $\mathrm{H}^1(\Gamma, G)$, since the inflation-restriction sequence is an exact sequence of pointed sets. Now for any $[z]$ in $\mathrm{H}^1(\Gamma, G)$ with corresponding twist G^z, there is an a commutative diagram of the form

$$\begin{array}{ccc} \mathrm{H}^1(\mathcal{E}, G^z) & \xrightarrow{\mathrm{res}_{G^z}} & \mathrm{H}^1(u, G^z) \\ {\scriptstyle \tau_z}\uparrow & & \uparrow{\scriptstyle \tau_z} \\ \mathrm{H}^1(\mathcal{E}, G) & \xrightarrow[\mathrm{res}_G]{} & \mathrm{H}^1(u, G^z) \end{array},$$

and τ_z induces a bijection between the fiber over $\mathrm{res}(z)$ and the kernel of $\overline{\mathrm{res}}$ (the main ingredient here is the explicit description of τ_z, together with the fact that u still acts trivially on G^z). But now, again by the inflation-restriction-sequence, this kernel is given precisely by $\mathrm{H}^1(\Gamma, G^z)$.

(i) *If F is p-adic, then both maps are bijective.*

(ii) *If $F = \mathbb{R}$, then the second map is bijective and the first map has trivial kernel (but possibly non-trivial fibers away from the neutral element).*

In particular, we have found a "natural" set that always surjects onto the set that classifies inner forms!

Proof. Let us sketch the proof: For the map $\mathrm{H}^1(\Gamma, \bar{G}) \to \mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$ we use that we have a Γ-equivariant decomposition $\bar{G} = G_{\mathrm{adj}} \times Z(G)/Z$, which induces a group isomorphism

$$\mathrm{H}^1(\Gamma, \bar{G}) \xrightarrow{\sim} \mathrm{H}^1(\Gamma, G_{\mathrm{adj}}) \times \mathrm{H}^1(\Gamma, Z(G)/Z),$$

and the map in question is the natural projection, which is surjective. If G is split, then $Z(G)/G$ is a split torus which has vanishing first cohomology, so the second map bijective in this case.

For the surjectivity of a, one uses that it sits in the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow \mathrm{H}^1(\Gamma, G) & \xrightarrow{\mathrm{inf}} & \mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G) & \xrightarrow{\mathrm{res}} & \mathrm{Hom}(u, \Gamma) \\ \| & & \downarrow a & & \downarrow \xi^* \\ \mathrm{H}^1(\Gamma, G) & \longrightarrow & \mathrm{H}^1(\Gamma, \bar{G}) & \longrightarrow & \mathrm{H}^2(\Gamma, Z) \end{array} \tag{3}$$

where ξ is the element in $\mathrm{H}^2(\Gamma, u)$ defining $\mathcal{E}^{\mathrm{rig}}$, and $\xi^*\colon \mathrm{Hom}(u, Z) \to \mathrm{H}^2(\Gamma, Z)$ is given by $\phi \mapsto \phi(\xi)$. One can show that ξ is surjective, using a version of the Tate-Nakayama isomorphism. In the abelian case, the diagram (3) can then be extended to the right in each row by $\mathrm{H}^2(\Gamma, G)$, and the four-lemma implies the claim. In the non-abelian case, the diagram (3) does not suffice to conclude, but we can reduce to this case: Let $R \subset G$ be a Levi subgroup and $S \subset R$ a fundamental maximal torus (a maximal torus whose dimension of the split component is as small as possible). Then $Z \subseteq S$, and we a diagram of the form

$$\begin{array}{ccc} \mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to S) & \longrightarrow & \mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G) \\ a\downarrow & & \downarrow \\ \mathrm{H}^1(\Gamma, \bar{S}) & \longrightarrow & \mathrm{H}^1(\Gamma, \bar{G}) \end{array}$$

We have seen above that the map $a\colon \mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to S) \to \mathrm{H}^1(\Gamma, \bar{S})$ is surjective. Moreover, $\bar{S} \subseteq \bar{R}$ is again a fundamental maximal torus, and in this case, the bottom-horizontal map $\mathrm{H}^1(\Gamma, \bar{S}) \to \mathrm{H}^1(\Gamma, \bar{G})$ is known to be surjective ([Kot86, Lem.10.2]).

Finally, lets argue why $a\colon \mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G) \to \mathrm{H}^1(\Gamma, \bar{G})$ has trivial kernel: We have already seen that the map $\mathrm{H}^1(\Gamma, \bar{G}) \to \mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$ is injective, so the kernel of a agress with the kernel of the composition

$$\mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G) \xrightarrow{a} \mathrm{H}^1(\Gamma, \bar{G}) \longrightarrow \mathrm{H}^1(\Gamma, G_{\mathrm{adj}}).$$

One then shows that this kernel coincides with the kernel of $\mathrm{H}^1(\Gamma, G) \to \mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$, which is now accesible via the long-exact sequence assoicated to $Z(G) \hookrightarrow G$, i.e. it is enough to see that $G_{\mathrm{adj}} \to \mathrm{H}^1(\Gamma, Z(G))$ is surjective. For this, it does in fact suffice that $T_{\mathrm{adj}}(F) \to \mathrm{H}^1(\Gamma, Z(G))$ is surjective for $T \subseteq G$ a maximal torus, which follows from $\mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$ being trivial. Now finally, in the p-adic case, we know that $\mathrm{H}^1(\Gamma, G) \to \mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$ is in fact a group homomorphism, so the kernel being trivial implies that it is in fact injective. □

4 Extending the Kottwitz isomorphism

The goal now is to extend the morphism $\alpha\colon \mathrm{H}^1(\Gamma, G) \to \pi_0(Z(\widehat{G})^\Gamma)^D$ to also obtain a more manageable description of $\mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G)$.

Construction 4.1. Recall that we have the category $\mathcal{R}$ consisting of maps $[Z \to G]$, where G is a reductive group over F and Z is a finite multiplicative group.

Theorem 4.2. *There is an isomorphism*

$$\iota\colon \bar{Y}_{+,\mathrm{tor}} \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{ab}}(u \to \mathcal{E}^{\mathrm{rig}})$$

of functors $\mathcal{R} \to \mathrm{Set}$*, which lifts the morphism of functors* $\bar{Y}_{+,\mathrm{tor}} \to \mathrm{Hom}(u, -)^\Gamma$*. In the* p*-adic case, this functor is compatible with the Tate-Nakayama isormorphism in the sense that the diagram*

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma, G) & \xrightarrow{\mathrm{inf}} & \mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\sim} \\ \pi_0(Z(\hat{G})^\Gamma)^D & \hookrightarrow & \pi_0(Z(\hat{\hat{G}})^+)^D \end{array}$$

commutes for any finite central subgroup Z*, where* $Z(\hat{\hat{G}})^+$ *is the preimage of* $Z(\widehat{G})^\Gamma$ *inside* $Z(\hat{\hat{G}})$ *under the natural map* $\hat{\hat{G}} \to \widehat{G}$*.*

We will not prove this theorem in the seminar talk, but let us at least sketch what is going on: The first step is to construct a map in the case that the functors are restricted to $\mathcal{T}$ (i.e. the target in $[Z \to S]$ is a torus). Already the construction of the map is difficult — it involves the new notion of “unbalanced cup products” and a more explicit construction of the cocycle $\xi \in \mathrm{H}^2(\Gamma, u)$ that gave rise to the gerbe $\mathcal{E}^{\mathrm{rig}}$. Let us at least give the construction of $\bar{Y}_{+,\mathrm{tor}}$ in the case of $[Z \to S] \in \mathcal{T}$: Here, we set $\bar{S} = S/Z$ and then $\bar{Y} := \mathbb{X}_\bullet(\bar{S})$, $\bar{Y}_+ := \bar{Y}/I\bar{Y}$ for $I \subseteq \mathbb{Z}[\Gamma_{E/F}]$ with E/F an extension that splits S ($\bar{Y}_+$ is independent of this choice). This already suggests compatibility with the classical Tate-Nakayama duality as reviewed in the begining of the talk. Part of the proof is that we have the compatibility

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma, S) & \longrightarrow & \mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to S) \\ \uparrow & & \uparrow \\ \frac{\mathbb{X}_\bullet(S)}{I\mathbb{X}_\bullet(S)} & \longrightarrow & \bar{Y}_{+,\mathrm{tor}}([Z \to S]) \end{array}$$

Example 4.3. Let's continue Example 1.3. Now $\mathrm{SL}_{n\mathrm{der}} = \mathrm{SL}_n$ which has center given by μ_n. Recall that the Galois action on $\widehat{\mathrm{SL}_n}$ was trivial, so we have have $Z(\hat{\hat{G}}) = \mu_n$, which recovers what we already knew from Theorem 3.6.

Remark 4.4. It might be interesting to find an example where the unique quasi-split inner form of G is not split, and where the surjection

$$\mathrm{H}^1(u \to \mathcal{E}^{\mathrm{rig}}, Z(G_{\mathrm{der}}) \to G) \twoheadrightarrow \mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$$

from Theorem 3.6 has non-trivial fibers, but $\mathrm{H}^1(\Gamma, G) \to \mathrm{H}^1(\Gamma, G_{\mathrm{adj}})$ is not surjective (so we can't take unitary groups). I couldn't find one.

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